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Soliton and antisoliton resonant interactions

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Abstract. Using the Hirota formalism, Gibbon *et al* have shown that the evolution equation

$$u_t + u_x - u_{xxt} + (4u^2 + 2y_x z_t)_x = 0$$

with $u = y_t = z_x$, has the same solitary wave as the regularised long wave (RLW) equation

$$u_t + u_x - u_{xxt} + 6(u^2)_x = 0$$

and an exact two-soliton solution describing the elastic collision of two sech^2 profile solitary waves. Performing a more detailed analysis, we show that the two-soliton solution can also represent other processes like the resonant or the singular collision of two RLW-type solitary waves. The interaction type depends on the values of a characteristic parameter of the solution. We also prove that with the bilinear form associated with the evolution equation, a three-soliton solution of the Hirota type cannot exist.

We then study the equation

$$u_t + u_x - u_{xxt} + 3(u^2)_x + 6u_t z_x = 0$$

with $u = z_t$, associated with another bilinear form, which has the same solitary wave as the evolution equation. We prove the existence of N-soliton solutions, for arbitrary N, and analyse the behaviour of the solitonic solutions. As in the first case, the two-soliton solution can describe elastic, resonant or singular interaction of two RLW-type solitary waves. A remarkable feature of the resonant triad is that it always involves one positive and two negative waves. This triad corresponds to a fundamental vertex for the analysis of the elastic soliton-antisoliton interaction.

1. Introduction

The regularised long-wave (RLW) equation first suggested by Peregrine (1966) and Benjamin *et al* (1972)

$$u_t + u_x - u_{xxt} + 6(u^2)_x = 0 \tag{1}$$

has been introduced as an alternative model to the Korteweg-de Vries (KdV) equation for describing non-linear evolution of unidirectional long waves.

Equation (1) has a solitary wave of sech^2 profile

$$u_s = \tfrac{1}{4}kw\,\text{sech}^2[\tfrac{1}{2}(\theta + \tau)] \tag{2}$$

with $w = k(1-k^2)^{-1}$ where $|k| \neq 1$, $\theta = -kx + wt$ and τ is a real constant but, contrary to the KdV-type solitary wave, its amplitude is not always positive for all values of the parameter k.

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Furthermore, an analytic two-soliton solution, as currently understood, does not exist (Olver (1979) has proved that RLW has only three conservation laws).

Using the Hirota (1976, 1980) method, Gibbon *et al* (1976)† derived an equation which has a solitary wave of the same functional form as (2) as well as an exact two-soliton solution. This equation (MRLW I) is:

$$u_t + u_x - u_{xxt} + 4(u^2 + 2y_x z_t)_x = 0 \tag{3}$$

where $u = y_t = z_x$ and possesses solutions of the form

$$u = -\partial^2_{xt} \ln f(x, t). \tag{4}$$

The solitary wave solution corresponds to

$$f \equiv f^{(1)} = 1 + e^{\varphi} \qquad \varphi = \theta + \tau. \tag{5}$$

The two-soliton solution is associated with

$$f \equiv f^{(2)} = 1 + e^{\varphi_1} + e^{\varphi_2} + K_{12}\, e^{\varphi_1 + \varphi_2} \tag{6}$$

where $\varphi_i = \theta_i + \tau_i$, $\tau_i \in R$, $\theta_i = -k_i x + w_i t$, $w_i = k_i(1 - k_i^2)^{-1}$ and K_{12} is a function of k_1 and k_2 of the form

$$K_{12} = \frac{(k_1 - k_2)^2(1 + k_1 k_2)(3 + k_1 k_2 - k_1^2 - k_2^2)}{(k_1 + k_2)^2(1 - k_1 k_2)(3 - k_1 k_2 - k_1^2 - k_2^2)}. \tag{7}$$

Gibbon *et al* (1976) discuss this solution in a particular region of the (k_1, k_2) plane $(K_{12} > 0)$, where it describes the elastic collision of solitary waves with positive amplitudes (solitons). However, they do not mention that K_{12} can also be negative: it vanishes (or becomes infinite) on some particular curves (figure 1). When the vanishing of K_{12} is due to a resonance $(3 + k_1 k_2 - k_1^2 - k_2^2 = 0)$, the solution is found to represent

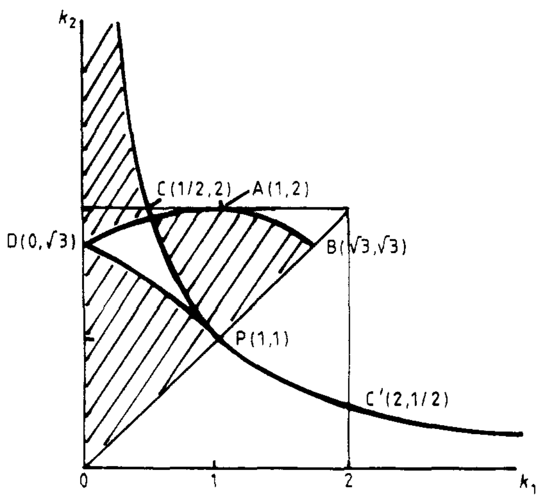

Figure 1. Inside the shaded regions, K_{12} is strictly positive. On the boundary $\widehat{DAB}$ belonging to the ellipse E_1, K_{12} is equal to zero, while on the curve $\widehat{DP}$ belonging to the ellipse E_2 and on the branch CPC′ of the hyperbole H_1, the inverse of K_{12} is zero. $E_1 \equiv 3 + k_1 k_2 - k_1^2 - k_2^2 = 0$. $E_2 \equiv 3 - k_1 k_2 - k_1^2 - k_2^2 = 0$. $H_1 \equiv 1 - k_1 k_2 = 0$.

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an inelastic process in which two solitary waves (one with a positive amplitude (soliton) and one with a negative amplitude (antisoliton)) fuse into a third wave with negative amplitude, or the reverse process (decay). The same two processes can be described by a two-soliton solution with $3-k_1k_2-k_1^2-k_2^2=0$ ($(K_{12})^{-1}=0$). We show that the MRLW I equation (3) does not possess three-soliton solutions of the Hirota type (§ 2).

In § 3, we propose a new equation (MRLW II) which has the same solitary wave solution as equations (1) and (3). We prove that this equation possesses N-soliton solutions for arbitrary N. It also possesses resonant solutions which, at $N=2$, describe the same inelastic processes as those obtained with the MRLW I equation (§ 4.1). They also occur on the elliptic curves $3 \pm k_1k_2-k_1^2-k_2^2=0$. These special solutions (and those obtained with equation (3)) are examples of resonant triads in one space dimension involving both solitons and antisolitons. They can be considered as a fundamental entity (vertex) for a detailed analysis of an elastic soliton-antisoliton collision (see § 4.2). We distinguish two basic processes (figure 5), each of which involves an 'intermediate antisoliton' with a lifetime proportional to $|\ln K_{12}|$. The possible occurrence of such an intermediate well (and the fact that it does not occur when a soliton collides with another soliton) might be of interest when MRLW solutions are compared with numerical results for the RLW equation (Abdulloev *et al* 1976, Eilbeck and McGuire 1977, Santarelli 1978, Courtenay Lewis and Tjon 1979, Bona *et al* 1980, 1985).

A general discussion of the resonance phenomenon for the MRLW II equation at arbitrary N shows that a regular solution cannot include other vertices than those obtained at $N=2$ (when $N \geqslant 3$, the solutions involve a resonant triad plus $N-2$ or $N-3$ 'spectator' solitary waves (see § 4.3)).

2. First modified RLW equation (MRLW I)

Given the linear part of equation (1), Gibbon *et al* (1976) consider the Hirota polynomial

$$F(D_t, D_x) = D_t(D_t + D_x - D_tD_x^2) \tag{8}$$

where the derivative operators D_t and D_x are defined by their action on an ordered pair of functions

$$\begin{aligned} D_x\tau_1(x, t)\tau_2(x, t) &= \lim_{\eta\to 0}\frac{\partial}{\partial\eta}\tau_1(x+\eta, t)\tau_2(x-\eta, t) \\ D_t\tau_1(x, t)\tau_2(x, t) &= \lim_{\eta\to 0}\frac{\partial}{\partial\eta}\tau_1(x, t+\eta)\tau_2(x, t-\eta). \end{aligned} \tag{9}$$

This polynomial operator acting on an ordered pair of two identical functions ($\tau_1=\tau_2=f$) yields the bilinear equation

$$D_t(D_t + D_x - D_tD_x^2)f(x, t).f(x, t) = 0 \tag{10}$$

that is,

$$-ff_{ttxx} + ff_{tt} + ff_{tx} - f_t^2 - f_tf_x + 2f_tf_{txx} + 2f_xf_{ttx} - 2f_{tx}^2 - f_{xx}f_{tt} = 0. \tag{11}$$

Equation (3) is obtained by making the transformation $f=\exp(q)$ in (11) and replacing q_{xt} by $-u$ in this result, as suggested by relation (4).

According to (9), the operator (8) acts in a simple way on an exponential or on ordered pairs of exponentials with arguments linear in x and t:

$$F(D_t, D_x)1.\exp(\theta_i) = F(D_t, D_x)\exp(\theta_i).1 = F(w_i, k_i)\exp(\theta_i)$$
$$F(D_t, D_x)\exp(\theta_i).\exp(\theta_j) = F(w_i - w_j, k_i - k_j)\exp(\theta_i + \theta_j). \tag{12}$$

It is then easy to verify that the one-soliton form (5) is a solution of equation (10). Indeed, the function

$$F(w, k) = w(w - k - wk^2) \tag{13a}$$

is identical to zero as a result of the 'dispersion' relation between w and k

$$w(k) = k(1-k^2)^{-1}. \tag{13b}$$

Using (12) and (13a), it is also easy to prove that the two-soliton form (6) is a solution of (10) provided that

$$K_{12} = -F(w_1 - w_2, k_1 - k_2)/F(w_1 + w_2, k_1 + k_2) \tag{14}$$

where $w_i = w(k_i)$, $i = 1, 2$. In the particular case (13a) and (13b), K_{12} takes the explicit form (7).

The existence of a three-soliton form

$$f \equiv f^{(3)} = 1 + \sum_{i=1}^{3}\exp(\varphi_i) + \sum_{1\leq i<j\leq 3} K_{ij}\exp(\varphi_i + \varphi_j) + K_{123}\exp(\varphi_1 + \varphi_2 + \varphi_3) \tag{15}$$

(where $\varphi_i = \theta_i + \tau_i$, $i = 1, 2, 3$, and K_{ij} are obtained from formula (7) by setting $1 = i$ and $2 = j$) as a solution of equation (10), is subject to two additional constraints to cancel the coefficients of $\exp(2\varphi_i + \varphi_j + \varphi_l)$ for $i \neq j \neq l$ and $\exp(\varphi_1 + \varphi_2 + \varphi_3)$ in the LHS of equation (10):

(i) $$K_{123} = K_{12}K_{13}K_{23} \tag{16}$$

(ii) $$\sum_{\substack{\{\mu_i = \pm 1\}\\ 1\leq i\leq 3}} F\left(\sum_{i=1}^{3}\mu_i w_i, \sum_{i=1}^{3}\mu_i k_i\right) \times \prod_{1\leq l<m\leq 3} F(\mu_l w_l - \mu_m w_m, \mu_l k_l - \mu_m k_m)\mu_l\mu_m = 0 \tag{17}$$

(the coefficients of other exponential terms are identically zero on account of relations (12) and (13a)).

The latter condition (17), which involves the actual form of F, cannot be satisfied when $F(w, k)$ has the form (13a). Indeed, setting

$$I(k_1, k_2, k_3) = \left(\sum_{\mathrm{cycl}(123)} k_1(1-k_2^2)(1-k_3^2)\right)\left(\sum_{\mathrm{cycl}(123)}(1-k_1^2)(1-k_2k_3)\right) \times \prod_{1\leq i<j\leq 3}(1+k_ik_j)(k_i-k_j)(3+k_ik_j-k_i^2-k_j^2) \tag{18}$$

the LHS of relation (17) is equivalent to

$$\frac{k_1^2k_2^2k_3^2(k_1^2-k_2^2)(k_1^2-k_3^2)(k_2^2-k_3^2)}{[(1-k_1^2)(1-k_2^2)(1-k_3^2)]^6}\hat{I}(k_1^2, k_2^2, k_3^2) \tag{19}$$

where

$$\hat{I}(k_1^2, k_2^2, k_3^2) = I(k_1, k_2, k_3) + I(-k_1, k_2, k_3) + I(k_1, -k_2, k_3) + I(k_1, k_2, -k_3).$$

Using MACSYMA, it is found that in the (k_1, k_2, k_3) space

$$\hat{I}(k_1^2, k_2^2, k_3^2) = 4k_1^2k_2^2k_3^2(k_1^2-k_2^2)(k_1^2-k_3^2)(k_2^2-k_3^2)$$
$$\times(2k_1^2+2k_2^2+2k_3^2-k_1^2k_2^2-k_1^2k_3^2-k_2^2k_3^2-3)$$
$$\times(k_1^2k_2^2k_3^4+k_1^2k_2^4k_3^2+k_1^4k_2^2k_3^2-6k_1^2k_2^2k_3^2-k_1^4-k_2^4-k_3^4$$
$$-k_1^2k_2^2-k_1^2k_3^2-k_2^2k_3^2+6k_1^2+6k_2^2+6k_3^2-9).$$

3. Second modified RLW equation (MRLW II)

Another bilinear equation associated with the linear part of equation (1) is

$$D_x(D_t+D_x-D_tD_x^2)f(x,t).f(x,t)=0 \tag{20a}$$

which is equivalent to the following quadratic equation in f:

$$ff_{xx}+ff_{xt}-ff_{xxxt}+3f_xf_{xxt}-3f_{xx}f_{xt}+f_{xxx}f_t-f_x^2-f_xf_t=0. \tag{20b}$$

By taking relation (4) into account, we obtain for u the evolution equation (MRLW II)

$$u_t+u_x-u_{xxt}+3(u^2)_x+6u_tz_x=0 \qquad z_t=u. \tag{21}$$

The polynomial in w and k associated with the differential operator of equation (20a) is

$$F_{II}(w,k)=k(w-k-k^2w). \tag{22}$$

It follows from (12) and (13b) that equation (21) has (just as equation (3)) the same solitary wave as the RLW equation and two-soliton solution of the form (6) with

$$K_{12}=-\frac{F_{II}(w_1-w_2,k_1-k_2)}{F_{II}(w_1+w_2,k_1+k_2)}=\frac{(k_1-k_2)^2(3+k_1k_2-k_1^2-k_2^2)}{(k_1+k_2)^2(3-k_1k_2-k_1^2-k_2^2)}. \tag{23}$$

Furthermore, we shall prove that (21) possesses N-soliton solutions (for arbitrary $N>2$) of the form

$$u^{(N)}(x,t)=-\partial_{xt}^2\ln f^{(N)}(x,t)$$

with

$$f^{(N)}(x,t)=1+\sum_{i=1}^{N}\exp(\varphi_i)+\sum_{1\le i<j\le N}K_{ij}\exp(\varphi_i+\varphi_j)$$
$$+\sum_{1\le i<j<l\le N}K_{ij}K_{il}K_{jl}\exp(\varphi_i+\varphi_j+\varphi_l)+\ldots$$
$$+\left(\prod_{1\le i<j\le N}K_{ij}\right)\exp\left(\sum_{i=1}^{N}\varphi_i\right) \tag{24}$$

where $\varphi_i=-k_ix+w_it+\tau_i$, $w_i=k_i(1-k_i^2)^{-1}$, $\tau_i\in R$ and

$$K_{ij}=\frac{(k_i-k_j)^2(3+k_ik_j-k_i^2-k_j^2)}{(k_i+k_j)^2(3-k_ik_j-k_i^2-k_j^2)} \qquad 1\le i<j\le N.$$

In fact, we should mention that the bilinear equation (20) has already been considered by Hirota and Satsuma (1976) but in relation with the evolution equation for the field $r(x, t)$ linked to $f(x, t)$ by $r(x, t) = 2\partial^2_{xx} \ln f(x, t)$. Their equation

$$r_t + r_x - r_{xxt} - 3rr_t + 3r_x z_t = 0 \qquad r = -z_x \tag{25}$$

differs from (21) by its non-linear part; its solitary wave solutions have always a positive amplitude (no antisolitons). Hirota and Satsuma claimed that (25) possesses N-soliton solutions. As we have been unable to find a published proof, we shall now show that both (21) and (25) have N-soliton solutions.

The condition for the existence of a N-soliton solution

$$\begin{aligned} Q^{(n)}(k_1, k_2, \ldots, k_n) &= \sum_{\substack{\{\mu_i = \pm 1\} \\ 1 \leq i \leq n}} F\left(\sum_{i=1}^{n} \mu_i w_i, \sum_{i=1}^{n} \mu_i k_i\right) \\ &\quad \times \prod_{1 \leq i < j \leq n} F(\mu_i w_i - u_j w_j, \mu_i k_i - \mu_j k_j)\mu_i \mu_j \\ &= \prod_{i=1}^{n} [k_i^{n-1}/(1-k_i^2)^{n-1}] \hat{Q}^{(n)}(k_1, k_2, \ldots, k_n) \\ &= 0 \qquad \text{for} \qquad n = 1, 2, \ldots, N \end{aligned} \tag{26}$$

amounts to the condition

$$\begin{aligned} \hat{Q}^{(n)}(k_1, k_2, \ldots, k_n) &= \sum_{\substack{\{\mu_i = \pm 1\} \\ 1 \leq i \leq n}} \left(\sum_{i=1}^{n} \mu_i k_i\right) \left\{\left[\left(\sum_{i=1}^{n} \mu_i k_i\right)^2 - 1\right] \sum_{i=1}^{n} \mu_i w_i + \sum_{i=1}^{n} \mu_i k_i\right\} \\ &\quad \times \prod_{1 \leq i < j \leq n} (\mu_i k_i - \mu_j k_j)^2 (3 + \mu_i \mu_j k_i k_j - k_i^2 - k_j^2) \\ &= 0 \qquad \text{for} \qquad n = 1, 2, \ldots, N \end{aligned} \tag{27}$$

which is proved by the following mathematical induction.

The function $\hat{Q}^{(n)}$ has the properties:

(i) $\hat{Q}^{(n)}(k_1, k_2, \ldots, k_n)$ is a symmetric function of $k_1^2, k_2^2, \ldots, k_n^2$

(ii) $\hat{Q}^{(n)}(k_1, k_2, \ldots, k_n)|_{k_1=0} = \prod_{i=2}^{n} k_i^2(3 - k_i^2) \hat{Q}^{(n-1)}(k_2, k_3, \ldots, k_n)$

(iii) $$\begin{aligned} \hat{Q}^{(n)}(k_1, k_2, \ldots, k_n)|_{k_1=k_2} &= 24k_1^2(1-k_1^2) \prod_{j=3}^{n} (k_1^2 - k_j^2)^2 \\ &\quad \times [(3 - k_1^2 - k_j^2)^2 - k_1^2 k_j^2] \hat{Q}^{(n-2)}(k_3, k_4, \ldots, k_n) \end{aligned}$$

(iv) $$\begin{aligned} \hat{Q}^{(n)}(k_1, k_2, \ldots, k_n) &= (3 + k_1 k_2 - k_1^2 - k_2^2)(k_1 - k_2)^2 \hat{Q}_1^{(n)}(k_1, k_2, \ldots, k_n) \\ &\quad + (3 - k_1 k_2 - k_1^2 - k_2^2)(k_1 + k_2)^2 \hat{Q}_1^{(n)}(k_1, -k_2, \ldots, k_n) \end{aligned}$$

with

$$\begin{aligned} \hat{Q}_1^{(n)}(k_1, k_2, \ldots, k_n) &= 2 \sum_{\substack{\{\mu_i = \pm 1\} \\ 3 \leq i \leq n}} \left(k_1 + k_2 + \sum_{i=3}^{n} \mu_i k_i\right) \left\{\left[\left(k_1 + k_2 + \sum_{i=3}^{n} \mu_i k_i\right)^2 - 1\right]\right. \\ &\quad \times \left.\left(w_1 + w_2 + \sum_{i=3}^{n} \mu_i w_i\right) + k_1 + k_2 + \sum_{i=3}^{n} \mu_i k_i\right\} \\ &\quad \times \prod_{i=3}^{n} (k_1 - \mu_i k_i)^2 (k_2 - \mu_i k_i)^2 (3 + \mu_i k_1 k_i - k_1^2 - k_i^2) \\ &\quad \times (3 + \mu_i k_2 k_i - k_2^2 - k_i^2) \\ &\quad \times \prod_{3 \leq i < j \leq n} (\mu_i k_i - \mu_j k_j)^2 (3 + \mu_i \mu_j k_i k_j - k_i^2 - k_j^2). \end{aligned}$$

On the other hand, we see that on the curve $3-k_1k_2-k_1^2-k_2^2=0$, the following relations are satisfied:

(*a*) $w_1+w_2=w(k_1+k_2)$

(*b*) $$(k_1-\mu_ik_i)^2(k_2-\mu_ik_i)^2(3+\mu_ik_ik_1-k_1^2-k_i^2)(3+\mu_ik_ik_2-k_2^2-k_i^2)$$
$$=[(k_1+k_2)-\mu_ik_i]^2([k_i^2+(k_1+k_2)^2-3]^2-k_i^2(k_1+k_2)^2)$$
$$\times[-3+(k_1+k_2)^2+k_i^2-\mu_ik_i(k_1+k_2)] \qquad 3\leqslant i\leqslant n.$$

It follows from the above relations that (on the curve $3-k_1k_2-k_1^2-k_2^2=0$)

$$\hat{Q}^{(n)}(k_1,k_2,\ldots,k_n)|_{3-k_1k_2-k_1^2-k_2^2=0}=-(3+k_1k_2-k_1^2-k_2^2)(k_1-k_2)^2$$
$$\times\prod_{i=3}^{n}([k_i^2+(k_1+k_2)^2-3]^2-k_i^2(k_1+k_2)^2)\hat{Q}^{(n-1)}(k_1+k_2,k_3,\ldots,k_n). \tag{28}$$

The identity (27) is easily verified for $n=1$ and $n=2$:

$$\hat{Q}^{(1)}(k_1)=F(w_1,k_1)=0$$

$$\hat{Q}^{(2)}(k_1,k_2)=\frac{(1-k_1^2)(1-k_2^2)}{k_1k_2}[F(w_1+w_2,k_1+k_2)F(w_1-w_2,k_1-k_2)$$
$$-F(w_1-w_2,k_1-k_2)F(w_1+w_2,k_1+k_2)]=0.$$

We now assume that this identity holds for $n-1$ and $n-2$ ($n\geqslant3$). Then, by using the properties (i), (ii) and (iii) and the result (28), we obtain the factorisation:

$$\hat{Q}^{(n)}(k_1,k_2,\ldots,k_n)\left(\prod_{i=1}^{n}k_i^2\prod_{1\leqslant i<j\leqslant n}(k_i^2-k_j^2)^2(3-k_ik_j-k_i^2-k_j^2)\right.$$
$$\left.\times(3+k_ik_j-k_i^2-k_j^2)\right)^{-1} \tag{29}$$

which is also valid for the polynomial $\hat{D}^{(n)}$

$$\hat{D}^{(n)}(k_1,k_2,\ldots,k_n)=\hat{Q}^{(n)}(k_1,k_2,\ldots,k_n)\prod_{i=1}^{n}(1-k_i^2). \tag{30}$$

Thus, according to the form of $\hat{Q}^{(n)}$ (see relation (27)) and the factorisation property (29), we see that the degree of the polynomial $\hat{D}^{(n)}$ is bounded as follows:

$$2n(2n-1)<\text{degree }\hat{D}^{(n)}<2n^2+2. \tag{31}$$

These inequalities imply that $\hat{D}^{(n)}\equiv0$.

4. Resonant multisoliton interaction for MRLW II

If regular, the N-soliton solution (24) of (21) can describe two kinds of interactions (depending on the values of the parameters k_i).

(i) An ordinary multisoliton interaction (elastic interaction) involving an arbitrary number N of solitons or antisolitons.

(ii) A resonant interaction involving a resonant triad (one soliton and two anti-solitons) in the presence of $N-2$ (or $N-3$) 'spectator' solitons (antisolitons).

As has already been pointed out (Tajiri and Nishitani 1982, Hirota and Ito 1983, Lambert *et al* 1987), the resonant interaction of solitons in one space dimension occurs on the boundaries of the regularity domain of the solution.

4.1. *The two-soliton solution*

The regularity condition for the two-soliton solution given by the relations (4) and (6) is

$$K_{12}=\frac{(k_1-k_2)^2(3+k_1k_2-k_1^2-k_2^2)}{(k_1+k_2)^2(3-k_1k_2-k_1^2-k_2^2)}\geqslant 0. \tag{32}$$

The corresponding regions in the first quadrant of the (k_1, k_2) plane are shown in figure 2 (without any loss of generality, we assume $k_2>k_1>0$). Inside the shaded regions, the solution $u^{(2)}$ describes the elastic soliton-soliton collision (if $k_1<k_2<1$), antisoliton-antisoliton collision (if $1<k_1<k_2$) or soliton-antisoliton collision (if $k_1<1<k_2$). More particularly, we distinguish the following cases.

(i) If $k_2<1$ or $k_2>2$, the interaction between the two solitary waves is always elastic.

(ii) If $1<k_2<2$, the curves $\widehat{\mathrm{DAB}}$ and $\widehat{\mathrm{DP}}$, on which $K_{12}=0$ or $(K_{12})^{-1}=0$, are the boundaries of the regularity domain of the two-soliton solution.

Setting $k_{\mathrm{R}}^{\pm}=k_2\pm k_1$ and $w_{12}^{\pm}=w_2\pm w_1$, one can see, on the curve $\widehat{\mathrm{DP}}$, where

$$E_2(k_1, k_2)\equiv 3-k_1k_2-k_1^2-k_2^2=0 \tag{33}$$

that $w_{12}^{+}=w(k_{\mathrm{R}}^{+})$. This follows from the identity

$$w_{12}^{\pm}-k_{\mathrm{R}}^{\pm}-(k_{\mathrm{R}}^{\pm})^2w_{12}^{\pm}=\frac{\mp k_1k_2(k_2\pm k_1)(3\mp k_1k_2-k_1^2-k_2^2)}{(1-k_1^2)(1-k_2^2)}. \tag{34}$$

In the same way, it follows from this identity that on the curve $\widehat{\mathrm{DAB}}$, where

$$E_1(k_1, k_2)\equiv 3+k_1k_2-k_1^2-k_2^2=0 \tag{35}$$

we have $w_{12}^{-}=w(k_{\mathrm{R}}^{-})$.

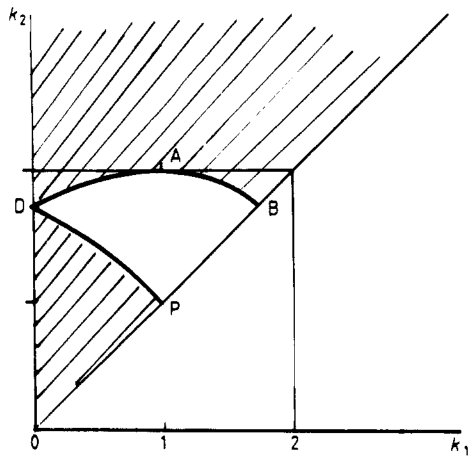

Figure 2. Inside the shaded regions, K_{12} is strictly positive. On the boundary $\widehat{\mathrm{DAB}}$, belonging to the ellipse E_1, K_{12} is equal to zero, while on the curve $\widehat{\mathrm{DP}}$, belonging to the ellipse E_2, the inverse of K_{12} is zero.

Thus, on the curve $\widehat{\mathrm{DP}}$, we have the relations

$$\begin{aligned} k_R^+ &= k_1 + k_2 \\ w_R^+ &= w_1 + w_2 \end{aligned} \tag{36}$$

while on the curve $\widehat{\mathrm{DAB}}$, we have

$$\begin{aligned} k_2 &= k_R^- + k_1 \\ w_2 &= w_R^- + w_1. \end{aligned} \tag{37}$$

Taking $\sqrt{3} \leqslant k_2 \leqslant 2$, the points situated on $\widehat{\mathrm{DAB}}$ correspond to $(k_1 = k_{2,\pm}, k_2)$ with

$$k_{2,\pm} = \tfrac{1}{2}\{k_2 \pm [3(4 - k_2^2)]^{1/2}\}. \tag{38}$$

We distinguish two possibilities.

$$\begin{array}{lll} \text{(i) } 0 < k_1 = k_{2,-} < 1 & \text{which implies} & 1 < k_R^- = k_{2,+} < \sqrt{3} \\ \quad\text{and the ordering (see figure 3)} & & v_1 > 0 > v_2 > v_R^- \end{array} \tag{39}$$

where $v_i = 1/(1 - k_i^2)$ stands for the phase velocity w_i/k_i.

$$\begin{array}{lll} \text{(ii) } 1 < k_1 = k_{2,+} < \sqrt{3} & \text{which implies} & 0 < k_R^- = k_{2,-} < 1 \\ \quad\text{and the ordering} & v_R^- > 0 > v_2 > v_1. & \end{array} \tag{40}$$

In the first case, the solution $u^{(2)} = -\partial_{xt}^2 \ln f^{(2)}(x, t)$ with

$$f^{(2)} = 1 + e^{\theta_1 + \tau_1} + e^{\theta_2 + \tau_2} \tag{41}$$

describes the decay of an antisoliton (with parameter $\sqrt{3} < k_2 < 2$) into a soliton and an antisoliton which travel in opposite directions with parameters $0 < k_{2,-} < 1$ and $1 < k_{2,+} < \sqrt{3}$ satisfying the relation $k_2 = k_{2,+} + k_{2,-}$ (see figure 4(*a*)). As $t \to -\infty$

$$u^{(2)}(x, t) \approx \frac{k_2 w_2}{4} \operatorname{sech}^2[\tfrac{1}{2}(\theta_2 + \tau_2)] \tag{42}$$

and as $t \to +\infty$

$$u^{(2)}(x, t) \approx \frac{k_1 w_1}{4} \operatorname{sech}^2[\tfrac{1}{2}(\theta_1 + \tau_1)] + \frac{k_R^- w_R^-}{4} \operatorname{sech}^2[\tfrac{1}{2}(\theta_R^- + \tau_2 - \tau_1)] \tag{43}$$

with $k_1 = k_{2,-}$ and $k_R^- = k_{2,+}$.

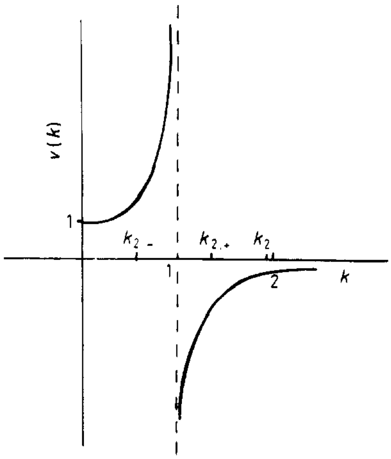

Figure 3. The curve represents the function $v(k) = (1 - k^2)^{-1}$ in terms of k, for $k > 0$. The value of k_2 is 1.95.

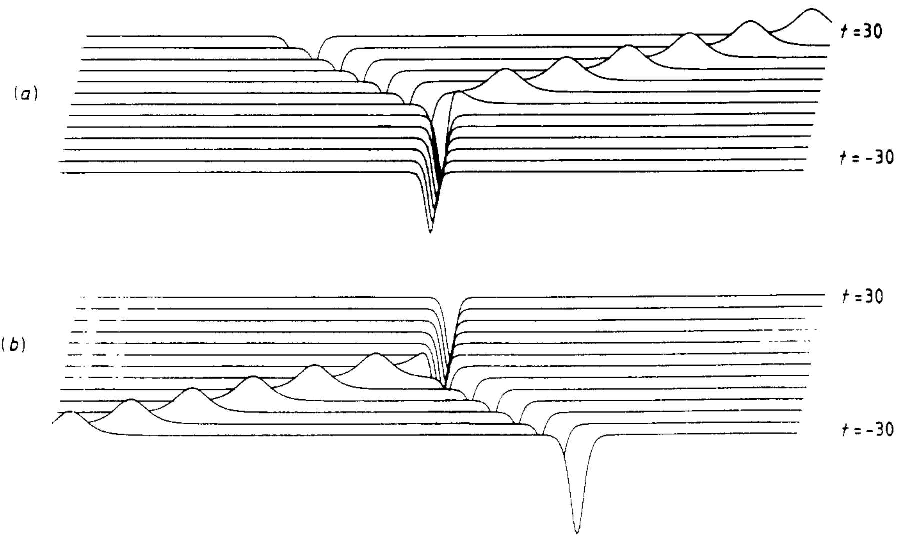

Figure 4. This figure represents the time evolution of the two-soliton solution for $K_{12}=0$ ($k_2=1.95$). In (a) ($k_1=k_{2,-}=0.59$), the process corresponds to the decay of an antisoliton (with amplitude $A_2=-0.34$) into a pair soliton-antisoliton (with amplitudes $A_1=0.13$ and $A_R=-0.54$). In (b) ($k_1=k_{2,+}=1.36$), the process corresponds to the fusion of a pair soliton-antisoliton (with amplitudes $A_R=0.13$ and $A_1=-0.54$) into an antisoliton ($A_2=-0.34$). The reference frame is defined by $\xi=x-v_2t$ ($-60\leq\xi\leq 60$).

In the second case, the solution $u^{(2)}$ describes the fusion of a soliton and an antisoliton (which travel in opposite directions with parameters $k_{2,-}$ and $k_{2,+}$) into an antisoliton k_2 (see figure 4(b)).

In both cases, the total mass $M=w_2$, which is conserved by the non-linear interaction, is negative.

Taking $1\leq k_2\leq\sqrt{3}$, the points situated on $\widehat{\mathrm{DP}}((K_{12})^{-1}=0)$ correspond to $(k_1=|k_{2,-}|, k_2)$ with

$$|k_{2,-}|=\tfrac{1}{2}\{-k_2+[3(4-k_2^2)]^{1/2}\}.$$

In this case, one has $0<k_1=|k_{2,-}|<1$, $\sqrt{3}<k_R^+=k_{2,+}<2$ and the ordering $v_1>0>v_R^+>v_2$.

It is easy to see that, with an appropriate choice of the phase parameters τ_1 and τ_2, the solution $u^{(2)}$, with $k_1=|k_{2,-}|$, also describes one of the above processes. Thus, by taking $\tau_1=\tau_1'$ and $\tau_2=\tau_2'-\ln K_{12}$ (τ_1' and τ_2' constant), one recovers the decay process of (42) and (43), whereas by taking $\tau_1=\tau_1'-\ln K_{12}$ and $\tau_2=\tau_2'$, one recovers the (reversed) fusion process.

4.2. Elastic processes near the boundaries of the regularity domain

The two inelastic processes shown in figure 4 correspond to solutions taken on the boundary of the regularity domain. Each such process can be represented by a diagram—a vertex joining two incoming (outgoing) lines with a third outgoing (incoming) line—which schematises this solution (contour map) in the (x, t) plane. These vertices turn out to be fundamental entities, as much as the soliton itself, for the interpretation of elastic soliton-antisoliton interactions (figure 5).

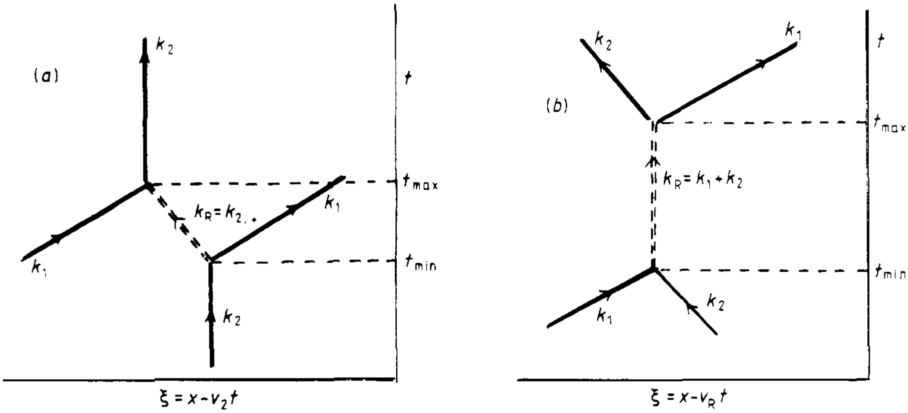

Figure 5. The time evolution of the elastic collision soliton-antisoliton is respectively schematised for (a) $K_{12} \to 0$ ($\xi = x - v_2 t$, $k_2 = 1.95$, $k_1 = 0.59$) and (b) $K_{12} \to \infty$ ($\xi = x - v_R t$, $k_2 = 1.36$, $k_1 = 0.59$). $\Delta t = t_{max} - t_{min}$ corresponds to the 'interaction time'.

For values of k_1, k_2 situated near the boundary $\widehat{\mathrm{DAB}}$ ($K_{12} \to 0$, $\sqrt{3} < k_2 < 2$, $k_1 \to k_{2,-} < 1$), the elastic collision of the two waves proceeds by the exchange of an intermediate antisoliton ($k_R = k_{2,+}$), the lifetime of which is proportional to $|\ln K_{12}|$. During this 'interaction time', three separate waves coexist: one well-shaped intermediate wave (deeper than the incoming antisoliton) which travels backwards between two bell-shaped waves of equal amplitude ($k_1 w_1 > 0$) moving forwards. A striking feature is that the two incoming waves are recovered as outgoing waves without ever colliding. They exchange identities.

For values of k_1, k_2 situated near the boundary $\widehat{\mathrm{DP}}$($(K_{12})^{-1} \to 0$, $1 < k_2 < \sqrt{3}$, $k_1 \to |k_{2,-}|$) the two incoming waves fuse together to form an intermediate antisoliton (in depth smaller than the incoming well-shaped wave) which, after a finite interaction time proportional to $\ln K_{12}$, decays into the original soliton-antisoliton pair. During the interaction time, there is only one (intermediate) wave.

4.3. *The N-soliton regular solutions*

The only regular N-soliton solutions (24) are those for which at most one parameter k_i is situated between 1 and $\sqrt{3}$ (all the k_i being ordered and positive).

Assuming we have the following ordering:

$$k_N > k_{N-1} > \ldots > 2 > k_l > k_{l-1} > \sqrt{3} > k_{l-2} > 1 > k_{l-3} > \ldots > k_1 \tag{44}$$

and setting

$$k_{i,\pm} = \tfrac{1}{2}\{k_i \pm [3(4 - k_i^2)]^{1/2}\} \qquad 1 \leqslant i \leqslant N \tag{45}$$

one has (see figure 6(a))

$$\sqrt{3} > k_{l-1,+} > k_{l,+} > 1 > k_{l,-} > k_{l-1,-}$$

and we must further have, to guarantee the regularity of the N-soliton solution ($K_{ij} \geqslant 0$ for $1 \leqslant i < j \leqslant N$), that (see figure 6($b$))

$$k_{l-2} \geqslant k_{l-1,+} \tag{46}$$

$$k_{l-3} \leqslant |k_{l-2,-}| \leqslant k_{l-1,-}. \tag{47}$$

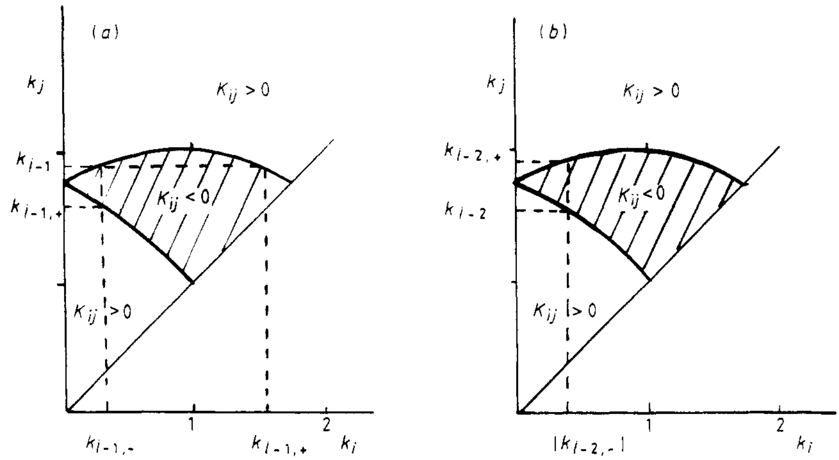

Figure 6. The regions in the plane k_i, k_j ($k_i < k_j$) where K_{ij} is greater than zero are bounded by the two ellipses $E_1(k_i, k_j)$ and $E_2(k_i, k_j)$. (*a*) $\sqrt{3} < k_{l-1} < 2$. (*b*) $1 < k_{l-2} < \sqrt{3}$.

The condition (46) implies that

$$\sqrt{3} < k_{l-2,+} \leqslant k_{l-1}.$$

We then distinguish four possibilities of resonant solutions.

(i) If $k_{l-2} = k_{l-1,+}$ (and $k_{l-3} < k_{l-1,-}$) then $K_{l-2,l-1} = 0$ (resonance involving the parameters k_{l-1}, k_{l-2} and $k_{l-1,-}$).

(ii) If $k_{l-3} = k_{l-1,-}$ (and $k_{l-2} > k_{l-1,+}$) then $K_{l-3,l-1} = 0$ (resonance involving the parameters k_{l-1}, $k_{l-1,+}$ and k_{l-3}).

(iii) If $k_{l-3} = |k_{l-2,-}|$ (and $k_{l-2,+} < k_{l-1}$) then $(K_{l-3,l-2})^{-1} = 0$ (resonance involving the parameters $k_{l-2,+}$, k_{l-2} and k_{l-3}).

(iv) If $k_{l-2} = k_{l-1,+}$ and $k_{l-3} = k_{l-1,-}$ then $K_{l-2,l-1} = K_{l-3,l-1} = 0$ and $(K_{l-3,l-2})^{-1} = 0$ (resonance involving the parameters k_{l-1}, k_{l-2} and k_{l-3}).

In the first three cases, the solution decribes an inelastic process as shown in figure 4 (with an appropriate choice of the phases τ_{l-2} and τ_{l-3} when $(K_{l-3,l-2})^{-1} = 0$) in the presence of $N-2$ 'spectators'. The resonant triad involves two waves of the sequence (44) and a third exotic wave. In the last case, the solution also describes an inelastic process of the kind discussed at $N = 2$ (with an appropriate choice of the phases τ_{l-2} and τ_{l-3}) but this time in the presence of $N-3$ 'spectators'. The resonant triad involves three waves (k_{l-1}, k_{l-2} and k_{l-3}) of the sequence (44).

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